Eur. Phys. J. A **31**, 758–760 (2007)
DOI 10.1140/epja/i2007-10007-5

THE EUROPEAN PHYSICAL JOURNAL A

Special Article – QNP 2006

Goldstone boson counting in relativistic systems at finite density

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Received: 8 January 2007
Published online: 22 March 2007 –

Abstract. We study the effects of finite chemical potential on the pattern of symmetry breaking within the relativistic linear sigma model. In accordance with previous works we show that type-II Goldstone bosons may appear whose dispersion relation is quadratic in momentum in the long-wavelength limit. We show that their presence is tightly connected with nonzero densities of non-Abelian Noether charges, and formulate a general counting rule for the number of the Goldstone bosons. Working at tree level, we conclude with the discussion of the loop effects. Our results find an application in particular to cold dense quark matter, where a type-II Goldstone boson has been found, *e.g.*, in the phases with kaon condensation.

PACS. 11.30.Qc Spontaneous and radiative symmetry breaking

1 Introduction

Spontaneous symmetry breaking plays a key role in understanding vastly different physical phenomena in several branches of physics, ranging from current high-energy and particle physics to condensed matter. One of its general consequences is the existence of the so-called Goldstone bosons —soft fluctuations of the order parameter(s)— guaranteed by the Goldstone theorem [1,2]. The number and properties of the Goldstone bosons (in particular, their dispersion relations) are essential for the low-energy dynamics of a system with spontaneously broken symmetry. Moreover, they significantly affect the thermodynamical properties of the system such as the heat capacity or the transport coefficients.

As written in any textbook on relativistic quantum field theory, in Lorentz-invariant theories the number of Goldstone bosons associated with a spontaneously broken *internal* symmetry (the case of a broken spacetime symmetry is treated, for instance, in ref. [3]) is always equal to the number of broken symmetry generators. On the other hand, in Lorentz-noninvariant systems[1], the situation is more complicated. The basic result in this respect was achieved by Nielsen and Chadha [4]. They showed that under certain technical assumptions, the energy of the Goldstone boson is proportional to some power of momentum in the long-wavelength limit. The Goldstone boson is then classified as type I, if this power is odd, or type II, if it is even, respectively. The Nielsen-Chadha counting rule states that *the number of type-I Goldstone bosons plus twice the number of type-II Goldstone bosons is greater or equal to the number of broken generators.*

In the past decade, the number of Goldstone bosons was proven to be connected with the possibility that some of the conserved (Noether) charges develop nonzero density in the ground state. In particular, Leutwyler analyzed spontaneous symmetry breaking in nonrelativistic systems within the framework of low-energy effective field theory [5]. He showed that nonzero density of a non-Abelian charge induces a term in the effective Lagrangian with a single time derivative, which, in turn, gives rise to a type-II Goldstone boson with a quadratic dispersion relation.

To summarize, the Nielsen-Chadha theorem clarifies the connection of the number of the Goldstone bosons and their dispersion relations. These are related, by Leutwyler's work, to the Noether charge densities. However, to the best of the author's knowledge, a direct connection of the Goldstone boson counting and the charge densities is still missing. A partial result in this respect was achieved by Schaefer *et al.* [6]: *The Goldstone boson counting is as usual, provided the densities of commutators of all pairs of broken generators vanish.*

In this contribution, we investigate the spontaneous symmetry breaking within the relativistic linear sigma model at finite chemical potential. This model was used to describe kaon condensation in the so-called Color-Flavor-Locked phase of dense quark matter [6,7]. Within this restricted framework, we are able to convert the theorem

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[1] In the following, these will be collectively called *nonrelativistic* in order to simplify the nomenclature. One should, however, keep in mind that this term may denote both intrinsically nonrelativistic systems and relativistic systems with Lorentz invariance explicitly broken, *e.g.*, by nonzero density.

of Schaefer *et al.* and show that the existence of type-II Goldstone bosons is unavoidable once some of the Noether charges develop nonzero density. Furthermore, we show that the Nielsen-Chadha inequality for the number of the Goldstone bosons is saturated. In the concluding section, we discuss the extension of our results to other relativistic systems at finite density.

2 Linear sigma model at finite chemical potential

2.1 Model with SU(2) × U(1) symmetry

We start with the model of Schaefer *et al.* [6], and Miransky and Shovkovy [7]. It is defined by the Lagrangian,

$$\mathcal{L} = D_\mu\phi^\dagger D^\mu\phi - M^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2, \tag{1}$$

where $D_\nu\phi = (\partial_\nu - i\delta_{\nu 0}\mu)\phi$. The scalar field ϕ transforms into the doublet representation of the global $SU(2)$ symmetry group, and μ is the chemical potential associated with the global $U(1)$ symmetry (particle number).

This model describes relativistic Bose-Einstein condensation: When $\mu > M$, the scalar field develops nonzero vacuum expectation value. Consequently, the $SU(2) \times U(1)$ symmetry of the Lagrangian (1) is spontaneously broken to its $U(1)$ subgroup (different from the original $U(1)$). Thus, *three* of the symmetry generators are spontaneously broken. However, only *two* Goldstone bosons appear, one type I and one type II. Their low-energy dispersion relations read

$$E = \sqrt{\frac{\mu^2 - M^2}{3\mu^2 - M^2}}|\mathbf{p}|, \qquad E = \frac{\mathbf{p}^2}{2\mu}, \tag{2}$$

respectively.

2.2 Properties of the type-II Goldstone boson

To get more insight into the nature of the type-II Goldstone boson, we now investigate the corresponding plane-wave solutions of the classical equations of motion. With the standard choice of the vacuum expectation value v, the scalar ϕ is reparametrized as

$$\phi = \frac{1}{\sqrt{2}} e^{i\pi_k\tau_k/v} \begin{pmatrix} 0 \\ v + H \end{pmatrix}.$$

The type-II Goldstone boson is then annihilated by the complex field $\psi = \frac{1}{\sqrt{2}}(\pi_2 + i\pi_1)$. The relevant bilinear part of the Lagrangian reads

$$\mathcal{L}_\psi = 2i\mu\psi^\dagger\partial_0\psi + \partial_\mu\psi^\dagger\partial^\mu\psi. \tag{3}$$

At the leading order of the power expansion in energy and momentum, it is evidently of the Schroedinger type, which is due to the fact that the field ψ carries nonzero charge of the unbroken $U(1)$ symmetry, generated by the matrix $\frac{1}{2}(1 + \tau_3)$.

The Lagrangian (3) describes a free particle with exact dispersion relation $E = \sqrt{\mathbf{p}^2 + \mu^2} - \mu$, whose low-momentum limit is given by eq. (2). The corresponding classical plane wave is simply $\psi = \psi_0 e^{-ip\cdot x}$. The $SU(2) \times U(1)$ symmetry of the Lagrangian (1) gives rise to four conserved currents, the isospin current and the particle number current, $j^\nu = -2\,\mathrm{Im}\,\phi^\dagger T\partial^\nu\phi + 2\mu\delta^{\nu 0}\phi^\dagger T\phi$, where $T = \{\boldsymbol{\tau}, 1\}$, respectively. For the two broken generators, τ_1 and τ_2, that create the type-II Goldstone boson, we find

$$j_1^\nu = +(p^\nu + 2\delta^{\nu 0}\mu)v\sqrt{2}\,\mathrm{Re}\,\psi,$$
$$j_2^\nu = -(p^\nu + 2\delta^{\nu 0}\mu)v\sqrt{2}\,\mathrm{Im}\,\psi.$$

It is apparent that the type-II Goldstone boson corresponds to an isospin wave, circularly polarized in the plane perpendicular to the vacuum density of the isospin. Note also that the other circular polarization corresponds to an excitation with a gap 2μ so that there is indeed a single Goldstone boson which couples to the two broken generators.

The unbroken $U(1)$ symmetry generates the current $j^\nu = 2(p^\nu + \delta^{\nu 0}\mu)|\psi|^2$. This uniform current proves that the isospin wave transfers the unbroken charge. In other words, the type-II Goldstone boson carries the unbroken charge, which seems to be a generic feature of type-II Goldstone bosons.

2.3 General bilinear Lagrangians

At tree level, the spectrum of the linear sigma model follows from the bilinear part of the Lagrangian upon a proper reparametrization of the scalar field. Once the chemical potential is introduced, this bilinear Lagrangian attains new terms with a single time derivative, which communicate the effects of Lorentz violation by the dense medium to the excitation spectrum. The generic form of the bilinear Lagrangian one encounters is (see ref. [8] for details)

$$\mathcal{L}_{\text{bilin}} = \frac{1}{2}(\partial_\mu\pi)^2 + \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}f^2(\mu)h^2 - g(\mu)h\partial_0\pi. \tag{4}$$

As long as at least one of the functions $f(\mu), g(\mu)$ is nonzero, the Lagrangian (4) describes a massive mode, with a mass gap $\sqrt{f^2(\mu) + g^2(\mu)}$, and a Goldstone boson with a dispersion relation

$$E^2 = \frac{f^2(\mu)}{f^2(\mu) + g^2(\mu)}\mathbf{p}^2 + \frac{g^4(\mu)}{[f^2(\mu) + g^2(\mu)]^3}\mathbf{p}^4 + \mathcal{O}(\mathbf{p}^6). \tag{5}$$

Equation (5) shows that when $f(\mu) \neq 0$, *i.e.*, the chemical potential mixes a Goldstone field with a Higgs field (as is the case of π_3 and H in the simple model (1)), one finds the expected result: One massive and one massless excitation. On the other hand, mixing of two Goldstone fields (such as π_1 and π_2 above) gives rise to just *one* Goldstone boson, which is type-II —its dispersion relation is quadratic at low momentum.

2.4 Model with arbitrary symmetry

In ref. [8] we analyzed Bose-Einstein condensation in the linear sigma model with an arbitrary symmetry breaking pattern. The generic Lagrangian reads $\mathcal{L} = D_\mu \phi^\dagger D^\mu \phi - V(\phi)$. The covariant derivative, $D_\mu \phi = (\partial_\mu - iA_\mu)\phi$, involves a constant external field A_μ that accounts for the chemical potential.

To find the spectrum, one minimizes the potential $V(\phi)$ and determines the vacuum expectation value ϕ_0. Next the scalar field is reparametrized as $\phi = e^{i\Pi}[\phi_0 + H]$, where the matrix field Π factorizes out the Goldstone degrees of freedom, while H represents the radial (Higgs) modes. The resulting bilinear Lagrangian reads

$$\begin{aligned}\mathcal{L}_{\text{bilin}} = {} & \partial_\mu H^\dagger \partial^\mu H - V_{\text{bilin}}(H) \\ & -2\,\text{Im}\, H^\dagger A^\mu \partial_\mu H - 4\,\text{Re}\, H^\dagger A^\mu \partial_\mu \Pi \phi_0 \\ & +\phi_0^\dagger \partial_\mu \Pi \partial^\mu \Pi \phi_0 - \text{Im}\, \phi_0^\dagger A^\mu [\Pi, \partial_\mu \Pi]\phi_0, \end{aligned} \qquad (6)$$

where V_{bilin} is the bilinear part of the potential which depends explicitly only on H.

The bilinear Lagrangian in eq. (6) contains three terms with a single derivative, proportional to the chemical potential. It can be shown that, with a proper choice of the basis of the symmetry generators, every excitation mode appears in exactly one of these terms so that the analysis of the simple two-field Lagrangian (4) applies. The most notable result is that the Goldstone-Goldstone mixing term in the last line of eq. (6), which according to eq. (5) gives rise to type-II Goldstone bosons, is proportional to the ground-state expectation value of the commutator of two broken generators. This proves the assertion made in the Introduction that *a nonzero density of a commutator of two broken generators gives rise to one type-II Goldstone boson with a quadratic dispersion relation*. Moreover, it is also obvious that the Nielsen-Chadha inequality for the number of the Goldstone bosons is saturated. (The only exception to this saturation known to the author, is the case of phase transitions where the phase velocity of a type-I Goldstone boson may vanish, thus making it an "accidental" type-II Goldstone boson.)

3 Summary and outlook

In this contribution, we investigated spontaneous symmetry breaking within the relativistic linear sigma model at finite chemical potential. We clarified the connection of Goldstone boson counting and their dispersion relations with nonzero densities of the Noether charges. In particular, we proved that nonzero density of a commutator of two broken charges produces one type-II Goldstone boson with a quadratic dispersion relation. It should be stressed, however, that all the results were achieved at the classical, tree level. Nevertheless, in ref. [9] it was shown that they are not altered by the one-loop radiative corrections.

Besides the radiative corrections to the linear sigma model, it would also be desirable to extend the results to other relativistic systems at finite density. In such systems, the Lorentz invariance is broken in a very particular way by the presence of the dense medium. It is, however, manifest on the microscopic level and hence could serve to constrain the patterns of symmetry breaking and the properties of the Goldstone bosons. One could thus hopefully strengthen the Nielsen-Chadha counting rule for this restricted class of systems. Based on the results achieved so far, we conjecture that generally an equality holds instead of the inequality, and that the Goldstone boson dispersion relation is either linear or quadratic, depending on the Lagrangian. (Recall that Nielsen and Chadha just distinguish odd and even powers of momentum.)

A preliminary argument in this direction was already given [8]. It was shown that when the symmetry group is non-Abelian, the charge density itself may serve as an order parameter for symmetry breaking. As a consequence, a single Goldstone boson couples to the two broken charges whose commutator yields the order parameter. In fact, by a proper analysis one may even show that such a Goldstone boson then necessarily has a quadratic dispersion relation. Using Leutwyler's effective Lagrangian approach, the coefficients in the dispersion relation can be related to the amplitude for the annihilation of the Goldstone boson by the broken current. This leads to a convenient model-independent parametrization of the Goldstone boson dispersion relations, which can be used as a check on models of spontaneous symmetry breaking such as that of Nambu and Jona-Lasinio. This issue will be investigated in detail in our future work.

The author wishes to thank the organizers of the *International Conference on Quarks and Nuclear Physics 2006* for giving him an opportunity to present this material, and to J. Hošek for fruitful discussions. This work was supported in part by the Institutional Research Plan AV0Z10480505, and by the GACR grants Nos. 202/06/0734 and 202/05/H003.

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